BOUNDARY BEHAVIOUR OF HARMONIC FUNCTIONS ON HYPERBOLIC GROUPS

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Abstract. We consider random walks with finite support on non-elementary Gromov hyperbolic groups. For a given harmonic function on such a group, we prove that asymptotic properties of non-tangential boundedness and non-tangential convergence are almost everywhere equivalent. The proof is inspired from works of F. Mouton in the cases of Riemannian manifolds of pinched negative curvature and infinite trees. It involves geometric and probabilitistic methods.

1. Introduction

The study of non-tangential convergence of harmonic functions began in 1906 with P. Fatou [Fat06]. He states that a positive harmonic function on the unit disc of $\mathbb{R}^2$ admits at almost all point of the unit circle a non-tangential limit. It is true in a lot of general cases like euclidean half-spaces, trees ([Car72]), free groups ([Der75]), Riemannian manifolds of pinched negative curvature ([AnS85], [Anc87]), and Gromov hyperbolic graphs ([Anc90]). It is thus natural to study cases where the harmonic function is not necessarily positive. Fatou's conclusion is no longer true in this more general case, and one tried to give criteria for the harmonic function to admit non-tangential limit at a point of the boundary. In the case of the euclidean half space $\mathbb{R}^n \times \mathbb{R}_+^*$, A.P. Calderon and E.M. Stein ([Cal50a], [Cal50b], [Ste61]) proved that for a harmonic function u, the three following properties are equivalent for almost all point θ of the boundary:

- the function u is non-tangentially convergent at θ
- the function u is non-tangentially bounded at θ
- the area integral $\int_{\Gamma_\theta} |\nabla u(x,y)|^2 y^{1-n} dxdy$ is finite (for all Γ^θ where Γ^θ is a non-tangential cone).

By probabilistic methods, J. Brossard proved the same result in 1978 [Bro78]. A. Koranyi remarked that hyperbolic spaces provide a more natural setting for this study. Indeed, several notions have simpler expressions in this case: the boundary becomes an ideal one, non-tangential cones become tubular neighbourhoods of geodesic rays... Following this remark, F. Mouton proved an analogous result for harmonic functions on Riemannian manifolds of pinched negative curvature [Mou94], and for harmonic functions on trees [Mou00]. We prove here a partial analogue for non-elementary hyperbolic groups: for a harmonic function (the notion of harmonicity is here relative to a random walk on a Cayley's graph of the

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hyperbolic group), non-tangential convergence is almost everywhere equivalent to non-tangential boundedness.

We introduce in section 2 the notions of random walks and harmonic functions in hyperbolic groups and in section 3 the boundary at infinity, which enables to state our main result in section 4. The section 5 is devoted to the conditioning of the random walk to exit at a fixed point of the boundary and to the proof of a stochastic result. In order to prove the non-tangential convergence criterion, we state geometric lemmas in section 6. We then prove the main result in section 7.

2. Harmonic functions on hyperbolic groups

We shall briefly introduce notions of hyperbolic groups, random walks, harmonic functions, Green functions. The reader can refer to [Mou00] and [GdlH90] for more details.

2.1. **Hyperbolic groups.** The notion of Gromov hyperbolicity was introduced in the 80's by M. Gromov [Gro81]. A way to introduce it is the following:

Definition 2.1. *On a metric space (X, d), one defines the Gromov product of two points $x, y \in X$ with respect to $o \in X$ by*

$$(x, y)_o = \frac{1}{2}[d(x, o) + d(y, o) - d(x, y)]$$

For a real $\delta \geq 0$, a metric space X is said to be δ-hyperbolic if for all $x, y, z, o \in X$,

$$(x, z)_o \geq \min\{(x, y)_o, (y, z)_o\} - \delta$$

A metric space X is geodesic if for every pair of points x and y in X, there is a geodesic segment (not necessarily unique) joining x to y in X, *i.e.* the image of an isometric embedding of the real interval $[0, d(x, y)]$ into X which sends 0 to x and $d(x, y)$ to y.

The definition of Gromov hyperbolicity makes sense in all metric spaces. However, it has a good geometric interpretation when the space is supposed to be geodesic. A geodesic triangle consists of three points $x, y, z \in X$ together with geodesic segments α, β, γ (respectively from y to z, x to z and x to y) called the sides. A triangle is called η-thin for a real $\eta \geq 0$ if every point of a side is at distance at most η from the union of the two other sides. If a geodesic metric space X is δ-hyperbolic, then every geodesic triangle in X is 4δ-thin. Remark that the converse also holds, if every geodesic triangle in X is η-thin, then X is 3η-hyperbolic. The reader can keep in mind that the Gromov product $(x, y)_o$ can be seen as a rough measure of the distance between o and a geodesic segment joining x and y (see [GdlH90]). Precisely, if X is δ-hyperbolic and γ is a geodesic segment from x to y, then

$$d(o, \gamma) - 2\delta \leq (x, y)_o \leq d(o, \gamma) \tag{2.1}$$

One can now define a hyperbolic group in the sense of Gromov. Let S be a finitely generated group. Let us note $\mathcal{G}(S, Z)$ the Cayley's graph associated to a finite symmetric generator system Z, that is a graph which has vertices set S, and where two elements x and y are neighbours iff $x^{-1}y \in Z$. Thus, S is equipped with a neighbourhood relation, denoted by $\sim$. A path from x to y in S is a sequence $[x = x_0, x_1, ..., x_k = y]$ such that for all i, $x_{i-1} \sim x_i$. The integer k is the length

of the path. The group S carries an integer-valued metric, called the word metric: $d(x, y)$ is the minimum among all the lengths of the paths from x to y.

If the Cayley's graph $\mathcal{G}(S, Z_0)$ is hyperbolic for a finite generating system Z_0, then $\mathcal{G}(S, Z)$ is hyperbolic for all finite generating system Z (Remark that the hyperbolic constant depends on the generating system). One then says that S is hyperbolic.

A rich family of hyperbolic groups is provided by fundamental groups of compact Riemannian manifolds with negative curvature. Different results state that groups given by a finite presentation are generically hyperbolic (see for example [Gro87], [Cha95], [Ol92]).

2.2. **Random walks.** Let S be a hyperbolic group, Z a finite symmetric generating system and d the corresponding distance. Fix a finitely supported probability measure ν on S such that $supp(\nu)$ generates S as a semi-group and define on $\partial S \times \partial S$ the transition function $p(x, y) := \nu(\{x^{-1}y\})$. This function is *admissible* in the sense of Ancona ([Anc88]), which means that the following relations hold:

(1) $\exists c_0 > 0, \exists l \in \mathbb{N}^*$ such that $\forall x, y \in S$, $d(x, y) \leq 1 \Rightarrow \sum_{0 \leq j \leq l} p^j(x, y) \geq c_0$
(2) $\exists m_1 \in \mathbb{N}^*$ such that $\forall x, y \in S$, $p(x, y) > 0 \Rightarrow d(x, y) \leq m_1$

We then define the random walk on $\mathcal{G}(S, Z)$ as the Markov chain with states' space S and transition probabilities $p(x, y)$, $x, y \in S$. It is given by the family of random variables $(X_n)_{n \in \mathbb{N}}$ where X_n is the position at time n. We can choose the space $\Omega = \mathcal{C}(\mathbb{N}, S)$ of all infinite paths as probability space (then, $X_n(\omega) = \omega(n)$), equipped with the σ-algebra arising from the countable product of $\mathcal{P}(S)$. We will denote $(\mathbb{P}_z)_{z \in S}$ the law of this random walk, where $\mathbb{P}_z$ is the probability obtained when the walk starts from z. As usual, we note $(\mathcal{F}_n)_n$ the natural filtration of $(X_n)_n$.

The admissible conditions are geometric adaptedness properties of the transition function p to the structure of the graph $\mathcal{G}(S, Z)$. Under these hypothesis, the random walk is transient.

We can now state a property which will be useful in the following. For an almost surely finite stopping time T, we note Θ^T the map: $\Theta^T(\omega) = \omega(\cdot + T(\omega))$.

Lemma 2.2 (Strong Markov property)**.** *For a non-negative random variable F and an almost surely finite stopping time T one has*

$$\mathbb{E}_x[F \circ \Theta^T | \mathcal{F}_T] = u_F(X_T) \text{ where } u_F(y) = \mathbb{E}_y[F]$$

2.3. **Harmonic functions and the Green function.** In order to define harmonic functions and the Green function, we associate to the random walk a Laplace operator Δ which acts on functions $f : S \to \mathbb{R}$ by

$$\Delta f(x) = \mathbb{E}_x[f(X_1)] - f(x) = \sum_{y \in S} p(x, y) f(y) - f(x)$$

A function f is said to be harmonic if $\Delta f = 0$.

The Green function is thus defined on $S \times S$ by

$$G(x, y) := \sum_{n=0}^{\infty} \mathbb{P}_x[X_n = y] = \mathbb{E}_x[\sum_{n=0}^{\infty} \mathbf{1}_{\{X_n = y\}}].$$

It can be seen by the Markov property that the function $G(\cdot, y)$ is harmonic on $S \setminus \{y\}$.

A fundamental property of random walks on graphs is the following:

Lemma 2.3 (Martingale property)**.** *Let z be a point of S and f a function on S. Then, the sequence of random variables*

$$M_n = f(X_n) - \sum_{k=0}^{n-1} \Delta f(X_k)$$

is a $(\mathcal{F}_n)$-martingale for the probability $\mathbb{P}_z$. In particular, $(f(X_n))_n$ is a martingale if f is harmonic.

3. Boundary at infinity

We are interested in non-tangential convergence, thus we need a notion of boundary. In fact, there are here three notions of boundary. To define them, we need to fix a base point o, but the compactifications below do not depend on the choice of o.

- The *geometric boundary* ∂S. Let us denote by E the set of sequences $(x_i)_i$ in S such that $\lim_{i,j\to\infty}(x_i, x_j)_o = +\infty$ and by $\partial S = E/\mathcal{R}$ the set obtained by factoring E with respect to the equivalence relation: $(x_i)_i\mathcal{R}(y_j)_j$ iff $\lim_{i,j\to\infty}(x_i, y_j)_o = +\infty$. An equivalent way to describe ∂S is via equivalence of geodesic rays (see [GdlH90]). One can extend the Gromov product to two points $x, y \in \partial S$ (resp. $x \in S$ and $y \in \partial S$ or the inverse) with
$$(x, y)_o = \sup \liminf_{i,j\to\infty}(x_i, y_j)_o \qquad (\text{resp. } (x, y)_o = \sup \liminf_{j\to\infty}(x, y_j)_o)$$
where the supremum is taken over all sequences $(x_i)_i$ in the class of x and $(y_j)_j$ in the class of y. For a real $r > 0$ and a point $x \in \partial S$, denote by $V_r(x) = \{y \in S \cup \partial S \,|\, (x, y)_o \geq r\}$. We equip $S \cup \partial S$ with the unique topology containing open sets of S and admitting the sets $V_r(x)$ with $r \in \mathbb{Q}^+$ as neighbourhood base at any $x \in \partial S$. It provides a compactification $\hat{S}$ of S (that is a compact Hausdorff space with countable base of the topology such that S is open and dense in $\hat{S}$). The compactification $\hat{S}$ can also be obtained as the completion of S in a good choice of a metric on S (see [Woe00]).
- The *Martin boundary.* One defines the Martin kernel by $K(x, y) = \frac{G(x,y)}{G(o,y)}$. The Martin compactification $\hat{S}$ is the unique smallest compactification of S for which all kernels $K(x, .)$, $x \in S$, extend continuously. The Martin boundary is $\hat{S}\backslash S$. A sequence $(y_i)_i \in S^{\mathbb{N}}$ converges to the Martin boundary if $d(o, y_i) \to \infty$ and $(K(., y_i))_i$ converges pointwise. Two such sequences are equivalent if their limit coincides at each point of S. It enables to represent non-negative harmonic functions by non-negative measures on this boundary (see [Woe00]).
- The *Poisson boundary* can be seen as the set of end points of the random walk. It is then the minimal subset of the Martin boundary which allows to represent bounded harmonic functions.

Results by Ancona ([Anc87]) and Kaimanovich ([Kai94]) prove that in the case of non-elementary hyperbolic groups (that is hyperbolic groups which are not finite extensions of $\{0\}$ or $\mathbb{Z}$) and with admissible conditions on the transition function, these three compactifications coincide. In the following, we will assume that it is true. We can thus denote ∂S the boundary. There is a ∂S-valued random variable X_∞ such that the random walk $(X_n)_n$ converges $\mathbb{P}_z$-almost surely to X_∞ for all

$z \in S$ (see [Woe00]). When dealing with harmonic functions and random walks, there is a natural family of measures on ∂S called the harmonic measures $\mu_z, z \in S$. There is two ways to define these measures. The first one is as the exit law of the random walk starting from z. The second one is by solving the Dirichlet problem at infinity. The different measures μ_z are equivalent, so we can define a notion of μ-negligeability. These measures allow to represent bounded harmonic functions by the Poisson formula (see [Woe00] and lemma 5.3).

4. Main result

Setting: *We fix now a non-elementary hyperbolic group S, a finite symmetric generating system Z and a finitely supported probability measure ν on S such that supp(ν) generates S as a semi-group.*

We will note d the distance arising from Z, δ the hyperbolicity constant, c_0 and m_1 the admissibility constants and $o \in S$ a base point.

Let us now define the non-tangential notions. If $c > 0$ and $\theta \in \partial S$, denote by

$$\Gamma_c^\theta := \{x \in S \,|\, \exists \gamma \text{ geodesic ray from } o \text{ to } \theta \text{ such that } d(x,\gamma) < c\}$$

the non-tangential tube of radius c and vertex θ. A function u converges non-tangentially at θ if, for all $c > 0$, $u(x)$ has a limit as x goes to θ in Γ_c^θ. In the same way, the function is non-tangentially bounded at θ if, for all $c > 0$, u is bounded on Γ_c^θ.

Remark that ,these non-tangential notions do not depend on o, due to the alternative definition of ∂S by geodesic rays ([GdlH90]).

Theorem 4.1. *In the setting above, for a harmonic function u, both following properties are equivalent for μ-almost all $\theta \in \partial S$:*

1. *the function u converges non-tangentially at θ*
2. *the function u is non-tangentially bounded at θ*

Denoting

$$\mathcal{L}_c = \{\theta \in \partial S \,|\, \lim_{\substack{x\in\Gamma_c^\theta \\ x\to\theta}} u(x) \text{ exists and is finite } \}$$

$$\mathcal{N}_c = \{\theta \in \partial S \,|\, N_c^\theta(u) < \infty\} \text{ where } N_c^\theta(u) = \sup_{x\in\Gamma_c^\theta} |u(x)|$$

$$\mathcal{L} = \bigcap_{c>0} \mathcal{L}_c \qquad \text{and} \qquad \mathcal{N} = \bigcap_{c>0} \mathcal{N}_c$$

the theorem can be enounced by: $\mathcal{N} \approx \mathcal{L}$, where $\approx$ means that the two sets differ by a μ-negligeable set.

The proof of this result uses stochastic methods which will be explained in the next section.

5. Conditioning

By Doob's h-process, it's possible to condition the random walk to exit at a fixed point $\theta \in \partial S$ (see [Doo57] or [Dyn69]). The probability $\mathbb{P}_z^\theta$ on Ω thus obtained satisfies a strong Markow property and one has the following two properties:

Proposition 5.1. *Let F be a non-negative random variable. Then*

$$\mathbb{E}_z[F] = \int_{\partial S} \mathbb{E}_z^\theta[F] d\mu_z(\theta)$$

The probability $\mathbb{P}_z^\theta$ satisfies an asymptotic zero-one law: if an event A is asymptotic (*i.e.* it's invariant by the translation operator Θ) then, for all $\theta \in \partial S$, the map $z \mapsto \mathbb{P}_z^\theta(A)$ is constant on S and equals 0 or 1. The reader can refer to [Mou94], [Bro78] and [Dur84] for more details.

As said above, we shall use stochastic methods. Therefore, we define kind of analogues of non-tangential convergence or boundedness notions. Let u be a harmonic function. We define the set $\widetilde{\mathcal{N}^{**}}$ of trajectories ω such that $|u|$ is bounded on the thickened trajectory $\{y \in S | d(y,\omega) \leq m_1\}$,

$$\widetilde{\mathcal{N}^{**}} = \{\omega \in \Omega \,|\, \widetilde{N^*} < +\infty\} \text{ where } \widetilde{N^*} = \sup\{|u(y)| \,|\, y \in S, d(y,w) \leq m_1\}$$
$$\text{and } \mathcal{L}^{**} = \{\omega \in \Omega \,|\, \lim_{n\to\infty} u(X_n(\omega)) \text{ exists and is finite } \}$$

These two events are asymptotic, so by asymptotic zero-one law, quantities $\mathbb{P}_z^\theta(\widetilde{\mathcal{N}^{**}})$ and $\mathbb{P}_z^\theta(\widetilde{\mathcal{L}^{**}})$ have values 0 or 1 and do not depend on z. So we define the sets

$$\widetilde{\mathcal{N}^*} = \{\theta \in \partial S \,|\, \mathbb{P}_o^\theta(\widetilde{\mathcal{N}^{**}}) = 1\} \text{ and } \mathcal{L}^* = \{\theta \in \partial S \,|\, \mathbb{P}_o^\theta(\mathcal{L}^{**}) = 1\}$$

We say that u is *stochastically bounded* at $\theta \in \partial S$ if $\theta \in \widetilde{\mathcal{N}^*}$ and that u *converges stochastically* at θ if $\theta \in \mathcal{L}^*$.

Proposition 5.2. *Given a harmonic function u, one has the μ-almost inclusion*

$$\widetilde{\mathcal{N}^*} \mathrel{\tilde{\subset}} \mathcal{L}^*.$$

Proof. We will first prove the $\mathbb{P}_o$-almost inclusion $\widetilde{\mathcal{N}^{**}} \mathrel{\tilde{\subset}} \mathcal{L}^{**}$. For $m \in \mathbb{N}$, denote by $\widetilde{\mathcal{N}^{**}_m}$ the set of trajectories ω such that $|u|$ is bounded by m on the thickened trajectory $\{y \in S \,|\, d(y,\omega) \leq m_1\}$. By countable union, it is sufficient to prove that for all m, $\widetilde{\mathcal{N}^{**}_m} \mathrel{\tilde{\subset}} \mathcal{L}^{**}$. Denote by T_m the stopping time

$$T_m := \inf\{n \geq 0 \,|\, \max\{|u(y)| \,|\, y \in S, d(y,X_n) \leq m_1\} > m\}.$$

Remark that $\widetilde{\mathcal{N}^{**}_m} = \{T_m = +\infty\}$. Since u is harmonic, $(u(X_n))_{n\in\mathbb{N}}$ is a martingale for the probability $\mathbb{P}_o$ and thus $(u(X_{n\wedge T_m}))_n$ is a martingale too. With our choice of stopping time T_m, for all $n \in \mathbb{N}$, $|u(X_{n\wedge T_m})| \leq \max\{m, |u(X_o)|\}$, which implies by the martingale theorem that the martingale converges $\mathbb{P}_o$-almost surely. In particular, $(u(X_n))_n$ converges $\mathbb{P}_o$-almost surely on the event $\widetilde{\mathcal{N}^{**}_m}$. Finally, we proved that $\widetilde{\mathcal{N}^{**}_m} \mathrel{\tilde{\subset}} \mathcal{L}^{**}$ and since $\widetilde{\mathcal{N}^{**}} = \bigcup_m \widetilde{\mathcal{N}^{**}_m}$ we proved that $\widetilde{\mathcal{N}^{**}} \mathrel{\tilde{\subset}} \mathcal{L}^{**}$.

Using proposition 5.1, we have

$$0 = \mathbb{P}_o(\widetilde{\mathcal{N}^{**}} \setminus \mathcal{L}^{**}) = \int_{\partial S} \mathbb{P}_o^\theta(\widetilde{\mathcal{N}^{**}} \setminus \mathcal{L}^{**}) d\mu_o(\theta)$$

Then, $\mathbb{P}_o^\theta(\widetilde{\mathcal{N}^{**}} \setminus \mathcal{L}^{**}) = 0$ for μ-almost all $\theta \in \partial S$ and $\widetilde{\mathcal{N}^*} \mathrel{\tilde{\subset}} \mathcal{L}^*$. □

We end this section with the case of bounded harmonic functions ([Woe00], [Anc90]).

Lemma 5.3. *A bounded harmonic function u on S converges non-tangentially and stochastically for μ-almost all point $\theta \in \partial S$ and the unique function $f \in L^\infty(\partial S, \mu)$ such that*

$$u(x) = \int_{\partial S} f(\theta) d\mu_x(\theta) = \mathbb{E}_x[f(X_\infty)]$$

is μ-a.e. the non-tangential and stochastic limit of u.

6. Geometric lemmas

By use of hyperbolicity, we prove two lemmas, and deduce three corollaries. They will be used several times in the proof of theorem 4.1.

Lemma 6.1. *Given $\alpha > 0$, there exists a constant $C > 0$ such that, for all point $x \in S$ and all $\theta \in \partial S$,*

$$\mu_x(\{\xi \in \partial S \,|\, (\xi, \theta)_x \geq \alpha\}) \geq C$$

Proof. Remark that ∂S is a compact set and that the sets $W_r(\xi) := V_r(\xi) \cap \partial S = \{y \in \partial S \,|\, (\xi, y)_o \geq r\}$ provide a neigbourhood base at $\xi \in \partial S$. Denoting $\beta = \alpha + \delta$, we extract by compacity a finite covering $W_\beta(\xi_1), ..., W_\beta(\xi_k)$ of ∂S. As the support of the harmonic measure is the Poisson boundary of S which coincides with the geometric boundary, for all i, $\mu_o(W_\beta(\xi_i)) > 0$ and by finiteness, there exists a constant $C > 0$ such that for all i, $\mu_o(W_\beta(\xi_i)) \geq C$.

Now, fix $\theta \in \partial S$. There exists i such that $\theta \in W_\beta(\xi_i)$. For all $y \in W_\beta(\xi_i)$, $(y, \theta)_o \geq \min\{(y, \xi_i)_o, (\theta, \xi_i)_o\} - \delta \geq \alpha$ and thus

$$W_\beta(\xi_i) \subset W_\alpha(\theta).$$

Then $\mu_o(W_\alpha(\theta)) \geq C$. According to the group structure, the constant C can be taken independant on the base point's choice: the isometric action ρ of S on $\mathcal{G}(S, Z)$ by left translation extends to ∂S and for $x \in \partial S$,

$$\mu_x(\{\xi \in \partial S \,|\, (\xi, \theta)_x \geq \alpha\}) = \mu_o(W_\alpha(\rho(ox^{-1})(\theta))) \geq C.$$

□

For a borelian set $E \in \partial S$, we note $\Gamma_c(E) := \bigcup_{\theta \in E} \Gamma_c^\theta$.

Lemma 6.2. *For all $c > 10\delta$, there exists $\eta > 0$ such that for all borelian sets $E \subset \partial S$, one has*

$$\forall x \notin \Gamma_c(E),\, \mathbb{P}_x(X_\infty \notin E) \geq \eta$$

Proof. We begin by showing that for all $x, y, z \in S \cup \partial S$,

$$(x, y)_o \geq \min\{(x, z)_o, (y, z)_o\} - 2\delta. \tag{6.1}$$

To see it, choose, for $\epsilon > 0$, sequences in S with $x_i \to x$, $y_i \to y$, $z_i \to z$ and $z'_i \to z$ such that $\liminf_{i,j}(x_i, z_j)_o \geq (x, z)_o - \epsilon$ and $\liminf_{i,j}(z'_i, y_j)_o \geq (z, y)_o - \epsilon$. Then, take $\liminf_{i,j}$ through $(x_i, y_j)_o \geq \min\{(x_i, z_j)_o, (z_j, z'_i)_o, (z'_i, y_j)_o\} - 2\delta$ (note that $\liminf_{i,j}(z_j, z'_i)_o = +\infty$).

We will also need that for all $x \in S$, $\xi \in \partial S$, and all geodesic ray γ from o to ξ,

$$d(x, \gamma) - 2\delta \leq (o, \xi)_x \leq d(x, \gamma) + 2\delta. \tag{6.2}$$

By inequality (2.1), for all i, $d(x, \gamma([\![0, i]\!])) - 2\delta \leq (o, \gamma(i))_x \leq d(x, \gamma([\![0, i]\!]))$. Since $d(x, \gamma(i)) \to \infty$, for i big enough, $d(x, \gamma([\![0, i]\!])) = d(x, \gamma)$ and then for i big enough,

$$d(x, \gamma) - 2\delta \leq (o, \gamma(i))_x \leq d(x, \gamma).$$

Combining this inequality with the fact that if $(\xi_i)_i$ is a sequence such that $\xi_i \to \xi$, thus $(o, \xi)_x - 2\delta \leq \liminf_i (o, \xi_i)_x \leq (o, \xi)_x$ (see [Bri99]), we obtain inequality (6.2).

Let us come back to the lemma's proof. Fix $c > 10\delta$, E be a borelian set in ∂S and $x \notin \Gamma_c(E)$. Choose two arbitrary points $\xi_1, \xi_2 \in \partial S$ and a geodesic joining both

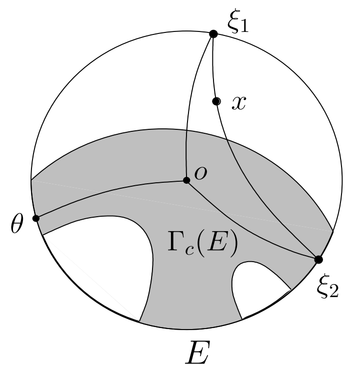

FIGURE 6.1. lemme 6.2

points (it exists by visibility (see [GdlH90])). Up to translation, one can assume without loss of generality that it contains x. Denoting γ_1 a geodesic ray from o to ξ_1 and γ_2 a geodesic ray from o to ξ_2, because all triangles are 4δ-thin, x is at distance at most 4δ from one of the two geodesic rays, γ_1 for example (see figure 6.1).

In order to use lemma 6.1, we will show that there exists a constant $\alpha > 0$ depending only on δ such that $\{\xi \in \partial S \,|\, (\xi, \xi_1)_x \geq \alpha\} \subset \partial S \setminus E$.

For $\theta \in E$, we want to bound uniformly from above the product $(\theta, \xi_1)_x$. Inequality (6.1) gives

$$\min\{(\theta, \xi_1)_x, (o, \theta)_x\} \leq (o, \xi_1)_x + 2\delta$$

By inequality (6.2), $(o, \xi_1)_x \leq d(x, \gamma_1) + 2\delta \leq 6\delta$, so $\min\{(\theta, \xi_1)_x, (o, \theta)_x\} \leq 8\delta$. Again by inequality (6.2), denoting γ a geodesic ray from o to θ, $(o, \theta)_x \geq d(x, \gamma) - 2\delta \geq c - 2\delta > 8\delta$ and $\min\{(\theta, \xi_1)_x, (o, \theta)_x\} = (\theta, \xi_1)_x \leq 8\delta$.

Therefore, $\{\xi \in \partial S \,|\, (\xi, \xi_1)_x \geq 9\delta\} \cap E = \emptyset$. By lemma 6.1, there exists $\eta > 0$ depending only on δ such that

$$\mathbb{P}_x(X_\infty \notin E) \geq \eta$$

.

□

Corollary 6.3. *Let E be a borelian set of ∂S, $x \in S$ and $c > 10\delta$. For μ-almost all $\theta \in E$, $\mathbb{P}_x^\theta$-a.s., the random walk "ends its life in $\Gamma_c(E)$" (Formally, for $\mathbb{P}_x^\theta$-almost all ω, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $X_n(\omega) \in \Gamma_c(E)$).*

Proof. Let $f_E(x) := \mathbb{P}_x(X_\infty \in E) = \mathbb{E}_x[\mathbf{1}_E(X_\infty)]$. As a consequence of lemma 5.3 of representation of bounded harmonic functions, for μ-almost all θ,

$$\forall x \in S,\, \mathbb{P}_x^\theta[\lim_{n\to\infty} f_E(X_n) = \mathbf{1}_E(\theta)] = 1$$

Because of lemma 6.2, there exists $\eta > 0$ such that

$$\forall x \notin \Gamma_c(E),\, f_E(x) \leq 1 - \eta$$

Thus for all $x \in S$ and for μ-almost all $\theta \in E$, $\mathbb{P}_x^\theta$-a.s., X_n is in $\Gamma_c(E)$ for n big enough. □

We will call spikes of a tube Γ_c^θ the sets $\Gamma_c^\theta \setminus B(o,R)$ for $R>0$.

Corollary 6.4. *Let $c>10\delta$ and E be a borelian set of ∂S. Then, for all $\theta\in\partial S$ such that $\lim\limits_{\substack{N.T.\\ x\to\theta}} \mathbb{P}_x(X_\infty\in E)=1$, $\Gamma_c(E)$ contains spikes of every tube with θ as vertex.*

In particular, it's true for μ-almost all $\theta\in E$ by the bounded harmonic function representation lemma 5.3.

Proof. Fix $\theta\in\partial S$ such that $\lim\limits_{\substack{N.T.\\ x\to\theta}} \mathbb{P}_x(X_\infty\in E)=\lim\limits_{\substack{N.T.\\ x\to\theta}} f_E(x)=1$ and let Γ_e^θ be a tube of vertex θ. By contradiction, assume that $\Gamma_c(E)$ doesn't contain any spike of this tube. Then, for each $R>0$, there exists $x\in{\Gamma_e}^\theta\setminus\Gamma_c(E)$ such that $d(o,x)>R$

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in ${\Gamma_e}^\theta\setminus\Gamma_c(E)$ such that $d(o,x_k)>k$. Since for all k, there exists a geodesic ray γ_k from o to θ such that $d(x_k,\gamma_k)<e$ and $\lim\limits_{k\to\infty} d(o,x_k)=+\infty$, $(x_k)_k$ converges to θ. The sequence $(x_k)_k$ lives in Γ_e^θ, then,

$$x_k \xrightarrow{N.T.} \theta \text{ and } \lim_{k\to\infty} f_E(x_k)=1.$$

Since $x_k\notin\Gamma_c(E)$, by lemma 6.2, $f_E(x_k)\le 1-\eta$, which gives the contradiction. □

The following corollary will not intervene later. However, it is interessant to remark that for a harmonic function, it is equivalent for μ-almost all point $\theta\in\partial S$ to be bounded on a tube of radius $c>10\delta$ and vertex θ and to be bounded on every tube of radius $c>0$ and vertex θ.

Corollary 6.5. *Given a harmonic function u, for all real $c>10\delta$ one has $\mathcal{N}_c\approx\mathcal{N}$.*

Proof. By definition, $\mathcal{N}\subset\mathcal{N}_c$. It is thus sufficient to show that $\mathcal{N}_c\,\tilde{\subset}\,\mathcal{N}$ for $c>6\delta$. Let $c>10\delta$ and denote

$$A_c^m=\{\theta\in\partial S\,|\,N_c^\theta(u)\le m\}.$$

As $\mathcal{N}_c$ is the countable union of the A_c^m, we need only show that $A_c^m\,\tilde{\subset}\,\mathcal{N}$ for all m. By definition of A_c^m, $|u|$ is bounded by m on $\Gamma_c(A_c^m)$. Using corollary 6.4, we obtain that for μ-almost all points $\theta\in A_c^m$, $\Gamma_c(A_c^m)$ contains spikes of all tube with θ as vertex. On these spikes, the function u is bounded, and therefore, by local finiteness, u is bounded on the tubes, which means that θ is in $\mathcal{N}$. Finally, $A_c^m\,\tilde{\subset}\,\mathcal{N}$ and therefore,

$$\mathcal{N}_c\,\tilde{\subset}\,\mathcal{N}.$$

□

7. Proof of the main result

Proof. As above, denote

$$A_c^m=\{\theta\in\partial S\,|\,N_c^\theta(u)\le m\}$$

Since $\mathcal{N}_c$ equals the countable union of the sets A_c^m, it is sufficient to prove that for all m and all $c>10\delta+m_1$, $A_c^m\,\tilde{\subset}\,\mathcal{L}_{c-m_1}$. Then, we will have $\mathcal{N}_c\,\tilde{\subset}\,\mathcal{L}_{c-m_1}$ and since for $c_1>c_2$, $\mathcal{L}_{c_1}\subset\mathcal{L}_{c_2}$, we will conclude that $\mathcal{N}\,\tilde{\subset}\,\mathcal{L}$.

Let $c>10\delta+m_1$. We shall first prove that $A_c^m\,\tilde{\subset}\,\mathcal{L}^*$. Applying corollary 6.3 to the borelian set A_c^m, we get: for μ-almost all point $\theta\in A_c^m$, $\mathbb{P}_z^\theta$-almost surely, $(X_k)_{k\ge0}$ ends its life in $\Gamma:=\Gamma_{c-m_1}(A_c^m)$. Let such a point θ. The key remark is that for all $x\in\Gamma$ and all $y\in S$ such that $d(x,y)\le m_1$, $|u(y)|\le m$. It implies

that for $\mathbb{P}_z^\theta$-almost all ω, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, for all $y \in S$ such that $d(y, X_n(\omega)) \leq m_1$, $|u(y)| \leq m$. By local finiteness, $\mathbb{P}_z^\theta$-almost surely, $\widetilde{N^*} < +\infty$. Thus, $\mathbb{P}_z^\theta(\widetilde{\mathcal{N}^{**}}) = 1$, $\theta \in \widetilde{\mathcal{N}^*}$ and hence $A_c^m \mathrel{\tilde{\subset}} \widetilde{\mathcal{N}^*}$. However, by proposition 5.2, $\widetilde{\mathcal{N}^*} \mathrel{\tilde{\subset}} \mathcal{L}^*$, so

$$A_c^m \mathrel{\tilde{\subset}} \mathcal{L}^*.$$

Let us now prove that $A_c^m \mathrel{\tilde{\subset}} \mathcal{L}_{c-m_1}$. As shown above, for μ-almost all $\theta \in A_c^m$, $\mathbb{P}_z^\theta$-almost surely, $(u(X_n))_n$ has a finite limit $l(\theta)$. It defines a function l on A_c^m. We use again corollary 6.3: for μ-almost all $\theta \in A_c^m$, $\mathbb{P}_z^\theta$-almost surely, X_n is in Γ for n big enough. Combining with the fact that $|u|$ is bounded by m on Γ, that gives $|l| \leq m$ on A_c^m.

The idea of the end of this proof, due to Brossard [Bro78], is to decompose u on Γ as a sum of three functions which will have non-tangential limits at almost all points of A_c^m.

We define the function

$$f(z) := \mathbb{E}_z[(l.\mathbf{1}_{A_c^m})(X_\infty)]$$

By the bounded harmonic functions representation lemma 5.3, f is a bounded harmonic function which converges non-tangentially at μ-almost all point $\theta \in \partial S$ to $(l.\mathbf{1}_{A_c^m})(\theta)$. Denote by τ the exit time of the set Γ and τ_k the exit time of $B(o,k)$. As u is bounded harmonic on the thickened set $\widetilde{\Gamma \cap B(o,k)} = \{y \in S \,|\, d(y, \Gamma \cap B(o,k)) \leq m_1\}$, which is a bounded set, $u(z) = \mathbb{E}_z[u(X_{\tau \wedge \tau_k})]$. If $\tau = +\infty$, $\mathbb{P}_z$-almost surely, $(X_n)_n$ converges to a point $X_\infty \in A_c^m$, so $\mathbb{P}_z$-almost surely, $(u(X_n))_n$ goes to $l(X_\infty)$. Denoting also $u(\theta) := l(\theta)$ for $\theta \in A_c^m$, we define u on A_c^m. Since $|u|$ is bounded by m on Γ, we can apply Lebesgue's theorem, and then we obtain

$$\forall z \in \Gamma, u(z) = \mathbb{E}_z[u(X_\tau)].$$

Decomposing the event $\{X_\infty \in A_c^m\}$ into the union $\{\tau < \infty \,|\, X_\infty \in A_c^m\} \cup \{\tau = \infty\}$ we obtain, for $z \in \Gamma$,

$$\begin{aligned} u(z) &= \mathbb{E}_z[u(X_\tau).\mathbf{1}_{\{\tau<\infty\}}] + \mathbb{E}_z[u(X_\infty).\mathbf{1}_{\{\tau=\infty\}}] \\ &= \mathbb{E}_z[u(X_\tau).\mathbf{1}_{\{\tau<\infty\}}] + \mathbb{E}_z[u(X_\infty).\mathbf{1}_{\{X_\infty \in A_c^m\}}] \\ &\quad -\mathbb{E}_z[u(X_\infty).\mathbf{1}_{\{X_\infty \in A_c^m\}}.\mathbf{1}_{\{\tau<\infty\}}] \end{aligned}$$

It is exactly the announced decomposition. Denoting $g(z) = \mathbb{E}_z[u(X_\tau).\mathbf{1}_{\{\tau<\infty\}}]$ and $h(z) = -\mathbb{E}_z[u(X_\infty).\mathbf{1}_{\{X_\infty \in A_c^m\}}.\mathbf{1}_{\{\tau<\infty\}}]$, it follows that $u = f + g + h$ on Γ.

We will prove that at almost all point θ in A_c^m, the functions g and h converge to zero staying in the tube $\Gamma_{c-m_1}^\theta$. Since u is bounded on $\widetilde{\Gamma} = \{y \in S \,|\, d(y, \Gamma) \leq m_1\}$, if $\tau < \infty$, then $|u(X_\tau)| \leq m$ and obviously

$$|g(z)| \leq m.\mathbb{P}_z(\tau < \infty).$$

In the same way, for almost all $\theta \in A_c^m$, $|u(\theta)| \leq m$, so we obtain easily by conditioning

$$|h(z)| \leq m.\mathbb{P}_z(\tau < \infty).$$

It is sufficient to prove that for almost all $\theta \in A_c^m$, $\mathbb{P}_z(\tau < \infty)$ goes to zero when z goes to θ staying in $\Gamma_{c-m_1}^\theta$. It's true because of lemma 6.2. Indeed, there exists $\eta > 0$ such that

$$\forall z \notin \Gamma, \mathbb{P}_z(X_\infty \notin A_c^m) \geq \eta$$

and this is in particular true for all $z \in \widetilde{\Gamma} \setminus \Gamma$. The strong Markov property implies that for all $z \in \Gamma$,

$$\begin{aligned}
\mathbb{P}_z(X_\infty \notin A_c^m) &= \mathbb{P}_z(\{X_\infty \notin A_c^m\} \cap \{\tau < \infty\}) \\
&= \sum_{i=1}^{\infty} \mathbb{P}_z(\{X_\infty \notin A_c^m\} \cap \{\tau = i\}) \\
&= \sum_{i=1}^{\infty} \mathbb{E}_z[\mathbb{P}_{X_\tau}(X_\infty \notin A_c^m).\mathbf{1}_{\{\tau = i\}}] \\
&= \mathbb{E}_z[\mathbb{P}_{X_\tau}(X_\infty \notin A_c^m).\mathbf{1}_{\{\tau < \infty\}}] \\
&\geq \eta.\mathbb{P}_z(\tau < \infty)
\end{aligned}$$

By lemma 5.3, for almost all $\theta \in A_c^m$, $\lim\limits_{\substack{N.T. \\ z \to \theta}} \mathbb{P}_z(X_\infty \notin A_c^m) = 0$. It follows that for almost all $\theta \in A_c^m$, $\mathbb{P}_z(\tau < \infty)$ goes to zero when z goes to θ staying in $\Gamma_{c-m_1}^\theta$. It follows that $A_c^m \,\widetilde{\subset}\, \mathcal{L}_{c-m_1}$ and the theorem is proved.

□

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